Teacher notes
Topic B

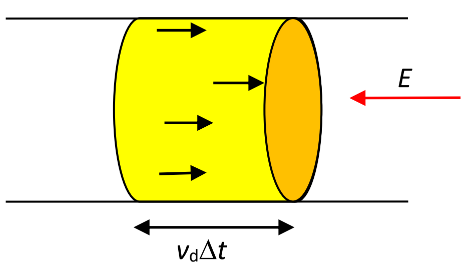
Why does the resistance of a metallic conductor increase with temperature?

In a metal, electrons move randomly at high speeds much like the molecules of a gas. Because the motion of the electrons is random electric charge is not transferred in any one direction and the electric current is zero. The average random speed of the electrons is very large. Using $\frac{1}{2}mc^2 = \frac{3}{2}kT$ we find at a temperature of 300 K:

$$c = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{9.1 \times 10^{-31}}} = 1.2 \times 10^5 \text{ m s}^{-1}$$

To establish a current, we need an electric field which will force electrons to move in the same direction. A very useful formula for current is $I = nAv_d e$ where v_d is the drift speed of the electrons, n is the number of electrons per unit volume and A is the cross-sectional area of the conductor. The drift speed is the average speed in the same direction that the electrons acquire because of the action of the electric field.

This can be derived very easily as follows: the shaded cylinder is part of a cylindrical conductor.



The length of the cylinder is $v_d \Delta t$ where v_d is the drift speed of electrons. In time Δt every electron in the shaded cylinder will pass through the cross-sectional area of the cylinder (shaded orange). The number of electrons in the shaded cylinder is $N = nV = nAv_d \Delta t$. The charge in the cylinder is then $\Delta Q = eN = enAv\Delta t$.

Therefore, the current is (charge through the cross-sectional area per unit time)

$$I = \frac{\Delta Q}{\Delta t} = \frac{enAv_d \Delta t}{\Delta t} = enAv_d .$$

If the conductor has length L and a potential difference V is established at its ends, the resistance will be $R=\frac{V}{I}$ but also $R=\rho\frac{L}{A}$. Hence

$\rho\frac{L}{A}=\frac{V}{enAv_{\mathrm{d}}}$ or $\rho=\frac{V}{enLv_{\mathrm{d}}}$. But $E=\frac{V}{L}$ is the electric field inside the conductor and so $\rho=\frac{E}{env_{\mathrm{d}}}$.

There are two speeds attached to the electrons. One is the random speed c which is very high as we saw. The second is the drift speed v_{d}. This is very small: imagine a length of 1 m of copper wire with a potential difference of 1 V at its ends. The resistivity of copper at 300 K is 1.7×10^{-8} Ω m and there are 8.56×10^{28} electrons per m^3. From $\rho=\frac{E}{env_{\mathrm{d}}}$ we find

$$v_{\mathrm{d}}=\frac{E}{en\rho}=\frac{1}{1.6\times10^{-19}\times8.56\times10^{28}\times1.7\times10^{-8}}\approx4\times10^{-3}\ \mathrm{m\ s^{-1}}=4\ \mathrm{mm\ s^{-1}}$$

To make further progress we need a model of how the electrons move in the wire. There is a very simple model called the Drude model (a classical model which is sufficient for our purposes here but not good in general). It assumes that each electron is accelerated by the electric field, picks up speed and then collides with lattice ions transferring the kinetic energy to the ions (hence the wire warms up). Each electron takes a different time t between one collision and the next. The average time in between successive collisions of an electron and a lattice atom is τ. The impulse provided by the electric force during a time t is $\vec{F}\Delta t=-(e\vec{E})t$. Impulse is the change in momentum and so (the bar denotes an average) $\bar{\vec{v}}=\bar{\vec{v}}_{random}-\frac{e\vec{E}}{m}\bar{t}=-\frac{e\vec{E}}{m}\tau$. This is because $\bar{\vec{v}}_{random}=0$; taking magnitudes gives:

$$v_{\mathrm{d}}=\frac{eE}{m}\tau$$

In other words, each electron has a *random* high velocity (of order $10^{5}\ \mathrm{m\ s^{-1}}$) and an average drift speed in the *same direction* (opposite to the electric field of order $10^{-3}\ \mathrm{m\ s^{-1}}$).

This means that

$$\rho=\frac{E}{env_{\mathrm{d}}}=\frac{E}{en\frac{eE\tau}{m}}$$

i.e.

$$\rho = \frac{m}{e^2 n\tau}$$

At 300 K,

$$\tau = \frac{m}{ne^2\rho} = \frac{9.1\times10^{-31}}{8.56\times10^{28}\times(1.6\times10^{-19})^2\times1.7\times10^{-8}} = 2.4\times10^{-14}\text{ s}$$

The distance travelled in this time is $d = c\tau = 1.2\times10^5\times2.4\times10^{-14} \approx 2.9\times10^{-9}$ m .

The relation $\rho = \frac{m}{e^2 n\tau}$ allows us to answer the original question, which was why the resistance increases as temperature increases. As the temperature increases the random speed of the electrons increases and so τ decreases. This means that the resistivity and hence resistance increases.